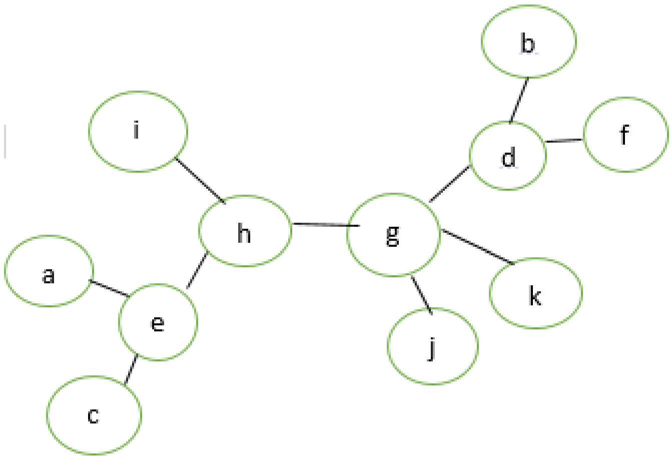
Some Basic Theorems on Trees

Tree:- A connected graph without any circuit is called a Tree. In other words, a tree is an undirected graph G that satisfies any of the following equivalent conditions:

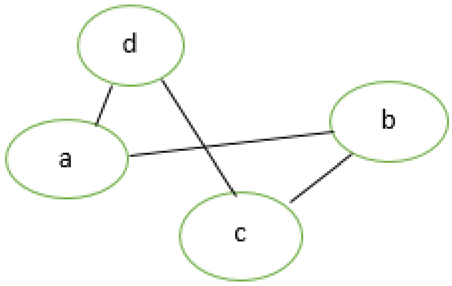
- Any two vertices in G can be connected by a unique simple path.
- G is acyclic, and a simple cycle is formed if any edge is added to G.
- G is connected and has no cycles.
- G is connected but would become disconnected if any single edge is removed from G.
- G is connected and the 3-vertex complete graph K3 is not a minor of G.

For Example:

- This Graph is a Tree:



- This Graph is not a Tree:



Some theorems related to trees are:

- **Theorem 1**: Prove that for a tree (T), there is one and only one path between every pair of vertices in a tree.

  Proof: Since tree (T) is a connected graph, there exist at least one path between every pair of vertices in a tree (T). Now, suppose between two vertices a and b of the tree (T) there exist two paths. The union of these two paths will contain a circuit and tree (T) cannot be a tree. Hence the above statement is proved.

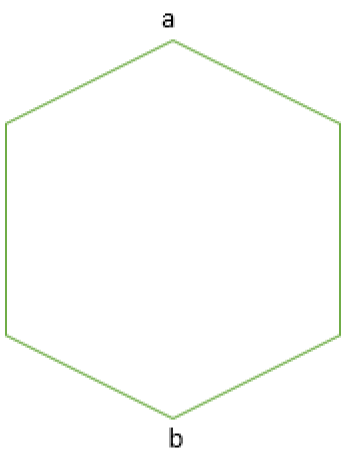


Figure 3: Tree(T)

- **Theorem 2:** If in a graph G there is one and only one path between every pair of vertices than graph G is a tree.

Proof: There is the existence of a path between every pair of vertices so we assume that graph G is connected. A circuit in a graph implies that there is at least one pair of vertices a and b, such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices. G cannot have any circuit. Hence graph G is a tree.

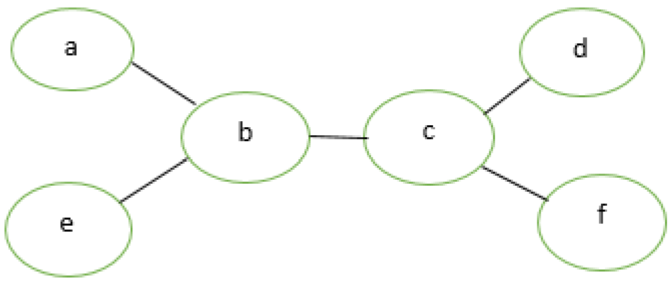


Figure 4: Given graph G

- **Theorem 3:** Prove that a tree with n vertices has (n-1) edges.

Proof: Let n be the number of vertices in a tree (T).
If n=1, then the number of edges=0.
If n=2 then the number of edges=1.
If n=3 then the number of edges=2.

Hence, the statement (or result) is true for n=1, 2, 3.

Let the statement be true for n=m. Now we want to prove that it is true for n=m+1.

Let e be the edge connecting vertices say Vi and Vj. Since G is a tree, then there exists only one path between vertices Vi and Vj. Hence if we delete edge e it will be disconnected graph into two components G1 and G2 say. These components have less than m+1 vertices and there is no circuit and hence each component G1 and G2 have m1 and m2 vertices.

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Now, the total no. of edges = (m1-1) + (m2-1) +1
                            = (m1+m2)-1
                            = m+1-1
                            = m.
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Hence for **n=m+1** vertices there are m edges in a tree (T). By the mathematical induction the graph exactly has n-1 edges.

Figure 5: Given a tree T

- **Theorem 4:** Prove that any connected graph G with n vertices and (n-1) edges is a tree.

  Proof: We know that the minimum number of edges required to make a graph of n vertices connected is (n-1) edges. We can observe that removal of one edge from the graph G will make it disconnected. Thus a connected graph of n vertices and (n-1) edges cannot have a circuit. Hence a graph G is a tree.

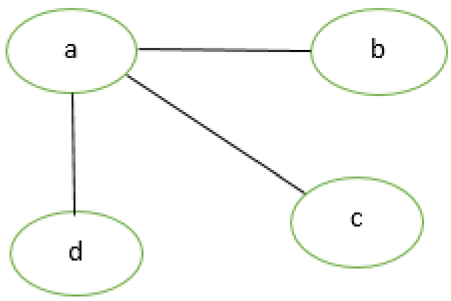

Figure 6: Graph G

- **Theorem 5:** Prove that a graph with n vertices, **(n-1)** edges and no circuit is a connected graph.

  Proof: Let the graph G is disconnected then there exist at least two components G1 and G2 say. Each of the component is circuit-less as G is circuit-less. Now to make a graph G connected we need to add one edge e between the vertices Vi and Vj, where Vi is the vertex of G1 and Vj is the vertex of component G2. Now the number of edges in **G = (n – 1)+1 =n.**

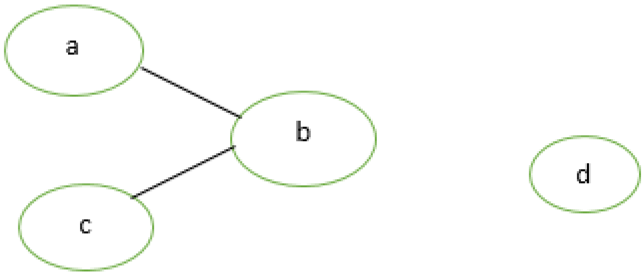

Figure 7: Disconnected Graph

Now, G is connected graph and circuit-less with n vertices and n edges, which is impossible because the connected circuit-less graph is a tree and tree with n vertices has **(n-1)** edges. So the graph G with n vertices, **(n-1)** edges and without circuit is connected. Hence the given statement is proved.

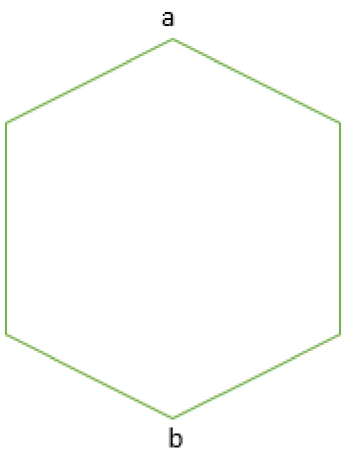

Figure 8:Connected graph G

- **Theorem 6:** A graph G is a tree if and only if it is minimally connected.

Proof: Let the graph G is minimally connected, i.e; removal of one edge make it disconnected. Therefore, there is no circuit. Hence graph G is a tree. Conversely, let the graph G is a tree i.e; there exists one and only one path between every pair of vertices and we know that removal of one edge from the path makes the graph disconnected. Hence graph G is minimally connected.

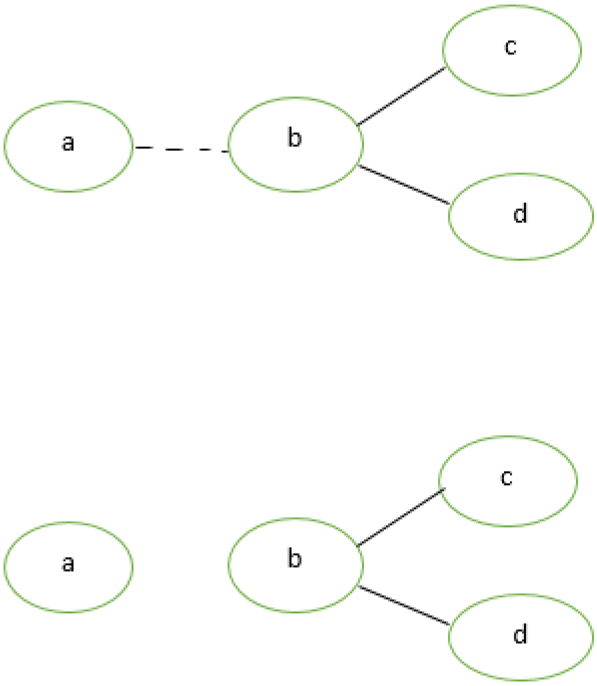

Figure 9: Minimally Connected Graph

- **Theorem 7:** Every tree with at-least two vertices has at-least two pendant vertices.

  Proof: Let the number of vertices in a given tree T is n and n>=2. Therefore the number of edges in a tree T=n-1 using above theorems.

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summation of (deg(Vi)) = 2*e
                       = 2*(n-1)
                       =2n-2
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  The degree sum is to be divided among n vertices. Since a tree T is a connected graph, it cannot have a vertex of degree zero. Each vertex contributes at-least one to the above sum. Thus there must be at least two vertices of degree 1. Hence every tree with at-least two vertices have at-least two pendant vertices.

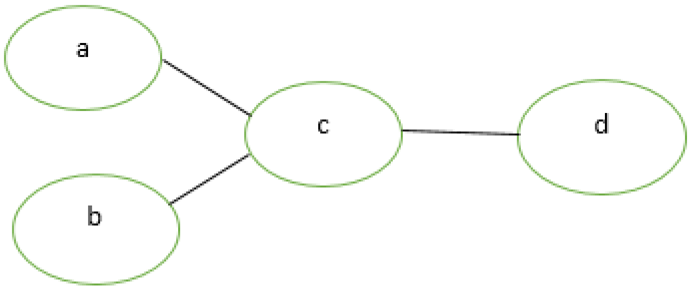

Figure 10: Here a, b and d are pendent vertices of the given graph

- **Theorem 8:** Show that every tree has either one or two centres.

  Proof: We will use one observation that the maximum distance max d(v, w) from a given vertex v to any other vertex w occurs only when w is pendant vertex.

  Now, let T is a tree with n vertices (n>=2)

  T must have at least two pendant vertices. Delete all pendant vertices from T, then resulting graph T' is still a tree. Again delete pendant vertices from T' so that resulting T" is still a tree with same centers.

  Note that all vertices that T had as centers will still remain centers in T'–>T"–>T"'–>….

  continue this process until the remaining tree has either one vertex or one edge. So in the end, if one vertex is there this implies tree T has one center. If one edge is there then tree T has two centers.

- **Theorem 9:** Prove the maximum number of vertices at level 'L' in a binary tree is 2^L, where L>=0.

  Proof: The given theorem is proved with the help of mathematical induction. At level 0 (L=0), there is only one vertex at level (L=1), there is only 2^1 vertices.

  Now we assume that statement is true for the level (L-1).

  Therefore, maximum number of vertices on the level (L-1) is $2^L - 1$. Since we know that each vertex in a binary tree has the maximum of 2 vertices in next level, therefore the number of vertices on the level L is twice that of the level L-1.

  Hence at level L, the number of vertices is:-

  $2^1 * 2^{(L-1)} = 2^L$.